SMII
Standard Model II
Exam 2019

Answer any Four questions

3 Hours

The numbers in square brackets in the right-hand margin indicate a provisional allocation of maximum possible marks for different parts of each question.

[Prof. Robert Thorne]

(Answer ANY FOUR questions)
Note: only four answers will be marked

1. (a) Starting from the Lagrangian for a fermionic field show that $\bar{\psi}$ satisfies the equation

$$\bar{\psi}\,(-i\overleftarrow{\not\partial} - m) = 0$$

where the arrow over $\not\partial$ implies the derivative acts on $\bar{\psi}$. Verify that this is equivalent to the usual manner of writing the Dirac equation in terms of $\psi(x)$. **[4]**

(b) For a photon with a mass m, the Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A^\mu A_\mu.$$

Define $F^{\mu\nu}$. Show that when this mass term is added Maxwell's equation $\partial_\mu F^{\mu\nu} = 0$ becomes instead $\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$. **[4]**

(c) Taking the divergence of this show that $\partial \cdot A = 0$, i.e. gauge fixing is automatic in this case. Hence show that $(\partial_\mu \partial^\mu + m^2)A^\nu = 0$. **[4]**

(d) The orthogonality property for the traces of generators may be written for any semi-simple Lie algebra as

$$\mathrm{tr}\{T_a^R T_b^R\} = T(R)\delta_{ab}$$

where $T(R)$ depends on the representation. Use this result and the commutation relations of the algebra

$$[T_a, T_b] = i f_{abc} T_c$$

to show that independent of R the structure constants are totally antisymmetric. **[4]**

(e) Prove the Jacobi identity

$$f_{abd}f_{dce} + f_{bcd}f_{dae} + f_{cad}f_{dbe} = 0$$

for these structure constants. **[4]**

2. (a) Consider a scalar field theory with a scalar with components ϕ_r. The potential for the field $V(\phi)$ is invariant under infinitesimal transformations

$$\delta\phi = iT_a\chi_a\phi\,, \quad a = 1, \ldots \dim G\,,$$

where T_a are the $\dim G$ generators of invariance group G in the representation defined by ϕ and χ_a are some infinitesimal parameters. The potential has a degenerate vacuum labelled by Φ_0. t_i are the generators of H, which is the stability group for $\phi_0 \in \Phi_0$, i.e.

$$t_i\phi_0 = 0\,, \quad i = 1, \ldots \dim H\,.$$

Choosing a basis for the generators such that

$$T_a = (t_i, T_{\hat{a}})\,,$$

with $T_{\hat{a}}$ orthogonal to t_i, prove, by expanding about the vacuum ϕ_0, that there are $\dim G - \dim H$ massless scalars. **[7]**

(b) A gauge theory for the group G is described by the Lagrangian,

$$\begin{aligned}\mathcal{L} &= -\tfrac{1}{4}\,F^{\mu\nu}{}_a F_{\mu\nu a} + \tfrac{1}{2}\,(D^\mu\phi)\cdot D_\mu\phi - V(\phi)\,,\\ F_{\mu\nu a} &= \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} - g\,c_{abc}A_{\mu b}A_{\nu c}\,, \quad D_\mu\phi = \partial_\mu\phi + ig\,A_{\mu a}T_a\phi\,,\end{aligned}$$

with $a = 1, \ldots \dim G$ and T_a anti-symmetric matrices, in the basis provided by real representation ϕ, representing the Lie algebra of G.
Suppose $V(\phi)$ is minimized at $\phi = \phi_0$ and that we fix the gauge invariance by imposing the 't Hooft gauge condition

$$\mathcal{L}_{\text{g.f.}} = -\tfrac{1}{2}\big(\partial^\mu A_{\mu a} + ig\phi\cdot(T_a\phi_0)\big)\big(\partial^\nu A_{\nu a} + ig\phi\cdot(T_a\phi_0)\big)\,.$$

If $\phi = \phi_0 + f$ derive the decoupled linearised equations of motion for the vector, scalar fields,

$$\partial^2 A_{\mu a} - g^2(T_a\phi_0)\cdot(T_b\phi_0)\,A_{\mu b} = 0\,, \quad \partial^2 f + \mathcal{M}f - g^2(T_a\phi_0)\,(T_a\phi_0)\cdot f = 0\,,$$

where $\mathcal{M}$ is a matrix determined by the second derivatives of $V(\phi)$ at $\phi = \phi_0$. **[8]**

(c) Show that the unbroken gauge group H has corresponding gauge fields which are massless and that the would-be Goldstone modes in f appear to be massive particles in addition to the remaining massive vector fields. **[5]**

3. (a) The Lagrangian for the gauge-scalar sector of the standard model may be written as ,

$$\mathcal{L} = -\frac{1}{4}\,\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu} - \frac{1}{4}\,G^{\mu\nu}G_{\mu\nu} + (D^\mu\phi)\cdot D_\mu\phi - \frac{1}{8}\lambda\big(\phi^2 - v^2\big)^2,$$

where

$$\begin{aligned} \mathbf{F}_{\mu\nu} &= \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + g\,\mathbf{A}_\mu\times\mathbf{A}_\nu \quad ,G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \\ (D^\mu\phi) &= (\partial^\mu + ig\tfrac{1}{2}\mathbf{A}^\mu(\mathbf{x})\cdot\sigma + i\tfrac{1}{2}g' B^\mu(x)), \end{aligned}$$

and $\mathbf{A}_\mu$ is the vector of $SU(2)$ gauge fields, B_μ is the $U(1)_Y$ gauge field and ϕ is a complex scalar doublet. σ_i are the Pauli matrices and g' may be written as $g\tan\theta_W$.

Explain why the scalar doublet can be written as

$$\phi(x) = \exp(-i\mathbf{n}(\mathbf{x}).\sigma + \mathbf{i n_3(x)})\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\begin{pmatrix} 0 \\ v + H(x) \end{pmatrix},$$

where $\mathbf{n} = (\mathbf{n_1}, \mathbf{n_2}, \mathbf{n_3})$, and why in unitary gauge we can eliminate the fields in $\exp(-i\mathbf{n}(\mathbf{x}).\sigma + \mathbf{i n_3(x)})$ completely. **[5]**

(b) Determine the simultaneous mass and charge eigenstates for the gauge fieldsby writing the scalar-boson interactions in terms of the physical fields Z_μ, $W^\pm_\mu$, and show that the photon field A_μ decouples from the scalar and is massless. Find also the mass of the scalar field and the the relationship between the masses m_W and m_Z. **[10]**

(c) Consider the part of the Lagrangian coupling the gauge bosons to the first generation lepton fields

$$\mathcal{L}_{\text{lept.}} = \bar{L}(x)i\gamma^\mu\partial_\mu L(x) + \bar{R}(x)i\gamma^\mu\partial_\mu R(x) - g\bar{L}\gamma^\mu\tfrac{1}{2}\sigma L.\mathbf{A}_\mu + g'(\tfrac{1}{2}\bar{L}\gamma^\mu L + \bar{R}\gamma^\mu R)\,B_\mu.$$

Show that the weak neutral current, i.e. the current coupling to the Z boson may be written as **[5]**

$$J^\mu_n = \tfrac{1}{2}\left[\overline{\nu}_e\gamma^\mu(1-\gamma_5)\nu_e - \overline{e}\gamma^\mu(1-\gamma_5 - 4\sin^2\theta_W)e\right] \;.$$

4. (a) In unitary gauge the scalar doublet in the Standard Model can be written as $\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}$, and has hypercharge $Y = \frac{1}{2}$. Show in full detail that the *conjugate* doublet $\phi^c(x) = i\sigma_2\phi(x)^*$ has the appropriate gauge transformation properties to give the up quark a mass via the interaction term

$$\mathcal{L}_{q\phi} = -\sqrt{2}[\bar{L}f^+\phi^c R_m^+ + \text{hermitian conjugate}],$$

where $L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$, $R^+ = u_R$ and f^+ is an arbitrary coupling. **[7]**

(b) When we consider three families the constant f^+ becomes a matrix and there is a similar matrix with elements f^- for the down-type quarks. The diagonalization of these matrices leads to the weak charged current having the form

$$J^\mu = \begin{pmatrix} \bar{u}, & \bar{c}, & \bar{t} \end{pmatrix} \gamma^\mu(1-\gamma^5)V_{CKM}\begin{pmatrix} d \\ s \\ b \end{pmatrix},$$

where V_{CKM} is a unitary matrix and the quarks are expressed in terms of mass eigenstates. Explain why V_{CKM} contains three independent angles and one complex phase and why for massless neutrinos there is no similar mixing in the charged current interactions. **[4]**

(c) The charged current interaction term in the Lagrangian is

$$\mathcal{L}_{cc} = -\frac{g}{2\sqrt{2}}(J^\mu W_\mu + J^{\mu\dagger}W_\mu^\dagger).$$

Show explicitly that under the combined parity and charge-conjugation transformations this Lagrangian is not invariant. **[6]**

(d) Briefly justify why, if we have right-handed, and hence massive, neutrinos we can add a "Majorana" mass term

$$\mathcal{L}_M = -m_M[(\bar{\nu}(x)_R)^C\nu(x)_R + \text{hermitian conjugate}]$$

as well as the standard "Dirac" mass term, whereas such a term is not allowed for other fermions. **[3]**

[Under parity transformations P

$$\psi(x) \to \gamma^0\psi(x_P) \qquad \bar{\psi}(x) \to \bar{\psi}(x_P)\gamma_0 \qquad W_\mu(x) \to W^\mu(x_P).$$

Under charge conjugation C

$$\psi(x) \to C\bar{\psi}^t(x) \qquad \bar{\psi}(x) \to -\psi^t(x)C^{-1} \qquad W_\mu(x) \to -W_\mu^\dagger(x),$$

where t denotes transpose and $C(\gamma^\mu)^tC^{-1} = -\gamma^\mu$.]

5. (a) Consider the decay process

$$\mu^-(p) \to e^-(k) + \overline{\nu}_e(q) + \nu_\mu(q').$$

Calculate the decay rate Γ for this process as explicitly as possible. You may start by assuming the matrix element for the decay is

$$\mathcal{M} = -\frac{G_F}{\sqrt{2}}\,\overline{u}_e(k)\gamma^\alpha(1-\gamma_5)v_{\nu_e}(q)\,\overline{u}_{\nu_\mu}(q')\gamma_\alpha(1-\gamma_5)u_\mu(p)\ ,$$

which leads to

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{G_F^{\,2}}{2} S_1^{\,\alpha\beta} S_{2\,\alpha\beta}\ ,$$

where, assuming the neutrinos have zero mass,

$$\begin{aligned} S_1^{\,\alpha\beta} &= \operatorname{tr}\left\{(\gamma.k+m_e)\gamma^\alpha(1-\gamma_5)\gamma.q\gamma^\beta(1-\gamma_5)\right\}\ ,\\ S_{2\,\alpha\beta} &= \operatorname{tr}\left\{(\gamma.p+m_\mu)\gamma_\beta(1-\gamma_5)\gamma.q'\gamma_\alpha(1-\gamma_5)\right\}\ . \end{aligned}$$

Show that we can make the simplification **[5]**

$$S_1^{\,\alpha\beta} S_{2\,\alpha\beta} = 256\, p.q\, k.q'\ .$$

(b) Demonstrate that this expression vanishes in a particular limit when $m_e \to 0$, and explain why this result is demanded by a conservation rule. **[3]**

(c) The decay rate may be written as

$$\Gamma = \frac{G_F^{\,2}}{8m_\mu \pi^5}\int \frac{\mathrm{d}^3k}{E_{\mathbf{k}}}\frac{\mathrm{d}^3q}{E_{\mathbf{q}}}\frac{\mathrm{d}^3q'}{E_{\mathbf{q'}}}\,\delta^4(p-k-q-q')\,p.q\,k.q'\ .$$

Show explicitly and in full detail that this reduces to **[6]**

$$\Gamma = \frac{G_F^{\,2}}{3m_\mu(2\pi)^4}\int \frac{\mathrm{d}^3k}{E_{\mathbf{k}}}\ \left(2p.(p-k)\,k.(p-k) + p.k\,(p-k)^2\right)\ .$$

(d) Using the approximation that $m_e/m_\mu = 0$ show that this becomes **[3]**

$$\Gamma = \frac{2G_F^{\,2}m_\mu}{3(2\pi)^3}\int_0^{\frac{1}{2}m_\mu} \mathrm{d}E\,E^2(3m_\mu - 4E)\ .$$

(e) Finally, evaluate the integral to obtain the result **[3]**

$$\Gamma_{\mu^-\to e^- + \overline{\nu}_e + \nu_\mu} = \frac{G_F^{\,2} m_\mu^5}{192\pi^3}\ .$$

6. (a) Consider the total cross-section for $e^-(p_1) + e^+(p_2) \to \gamma^\star(p_1 + p_2 = q) \to q(k_1) + \overline{q}(k_2)$. Using the result that

$$\sum_{\text{spins}} |\mathcal{M}|^2 = 32\frac{e^4 Q_q^2}{q^4}\big[(k_1\cdot p_1)(k_2\cdot p_2) + (k_1\cdot p_2)(k_2\cdot p_1)\big],$$

show that at lowest order

$$\frac{\mathrm{d}\sigma_{e^-e^+\to q\bar{q}}}{\mathrm{d}\Omega} = \frac{\alpha^2}{4q^2}\, Q_q^2\,(1+\cos^2\theta)\,,$$

where $\alpha = e^2/4\pi$, θ is the angle between the outgoing quark and the axis of the incoming electron and positron in the centre of mass frame, and Q_q is the fractional quark charge. Prove that integrating over the solid angle

$$\sigma_{e^-e^+\to q\bar{q}} = \frac{4\pi\alpha^2}{3q^2}\, Q_q^2\,.$$

You may assume that $\sqrt{q^2} \gg m_q, m_e$. **[8]**

(b) Explain why at leading order this means that (to a good approximation) **[4]**

$$\sigma_{e^-e^+\to \text{hadrons}} = \frac{4\pi\alpha^2}{3q^2}\, 3\sum_f Q_f^2\,.$$

(c) Discuss why beyond leading order the cross-section for quark-antiquark production is not a well-defined physical quantity, and how one may calculate the total hadron cross-section. **[4]**

(d) Beyond LO the cross-section may be written as

$$\sigma_{e^-e^+\to \text{hadrons}} = \frac{4\pi\alpha^2}{3q^2}\, 3\sum_f Q_f^2\, K(\alpha_s(\mu^2), q^2/\mu^2)\,,$$

where at $\mathcal{O}(\alpha_s^2)$

$$K(\alpha_s(\mu^2), q^2/\mu^2) = 1 + \frac{\alpha_s(\mu^2)}{\pi} + \frac{\alpha_s^2(\mu^2)}{\pi^2}\left(1.99 - 0.11 n_f - \pi\frac{\beta_0}{4\pi}\ln(q^2/\mu^2)\right).$$

One way of choosing the arbitrary scale μ is to demand that

$$\frac{dK(\alpha_s(\mu^2), q^2/\mu^2)}{d\ln\mu^2} = 0.$$

Using the renormalization group equation for the strong coupling

$$\frac{d\alpha_s}{d\ln\mu^2} = -\frac{\beta_0}{4\pi}\alpha_s^2,$$

where $\beta_0 = 11 - 2/3n_f$, and n_f is the number of quark flavours, determine the value of μ^2 this prescription imposes. **[4]**

[You may use

$$\sigma = \frac{1}{4F}\frac{1}{4}\sum_{\text{spins}}\int \frac{d^3\mathbf{k_1}}{(2\pi)^3 2E_{\mathbf{k_1}}}\frac{d^3\mathbf{k_2}}{(2\pi)^3 2E_{\mathbf{k_2}}}(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)|M|^2$$

where the flux factor $F = 4\sqrt{(p_1.p_2)^2 - m_1^2 m_2^2} = 2q^2$, where we let $m_1, m_2 \to 0$.]

END OF PAPER